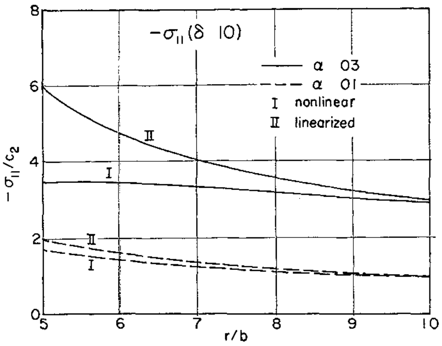

Fig 2 Distribution of radial stress

and, from Eqs (8) and (12),

$$\lambda_1\lambda_2^{-1}W_1 = -\frac{E_s}{1-\nu_s^2}\frac{h}{b}\frac{u_s}{b+u_s} \tag{14}$$

In this formulation of the problem, the corresponding magnitude of the internal pressure is obtained by calculating σ_{11} at $r = a$ Therefore, stresses and strains under a specified internal pressure are obtained by trial-and-error method in which Eqs (5) and (11) should be integrated for trial values of u until the computed internal pressure is found to be close enough to the specified value

In the present analysis, it is also possible to include certain nonhomogeneous materials for which the strain-energy function W depends on r explicitly, and not only through λ_1 and λ_2 This can be taken into account during the differentiation with respect to r in Eq (10)

The following form of W has been considered purely for the purpose of illustrating the presented method of analysis; the strain-energy function fails to reproduce certain fundamental physical relations such as the $\lambda_1 - \lambda_2$ relation under simple extension observed in experiments[5]:

$$W = C_1J_1^2 + C_2J_2 \tag{15}$$

The linearized problem corresponding to Eq (15) has also been considered where the Lamé constants $\lambda = 2C_1$ and $u = C_2$ are used for the generalized Hooke's Law with $\epsilon = \partial u_r/\partial r$ and $\epsilon_\theta = u/r$

For numerical computations, the following quantities are introduced as parameters:

$$\alpha = \frac{u_s}{b} \qquad \delta = \frac{E_s}{1-\nu}\frac{h/b}{C_2} \qquad \rho = \frac{b}{a} \qquad \gamma = \frac{C_1}{C_2}$$

where $\rho = 2\ 0$, $\gamma = 2\ 0$, $\delta = 10$, and $\alpha = 0\ 01$ or $0\ 03$ are used for numerical integrations The choice of $\gamma = 2\ 0$ implies that the Poisson ratio for infinitesimal deformation is 0 4

The results are shown in Figs 2 and 3 For comparison, the linearized solutions are also plotted in the same diagrams with designation II whereas the curves corresponding to Eq (15) are designated by I

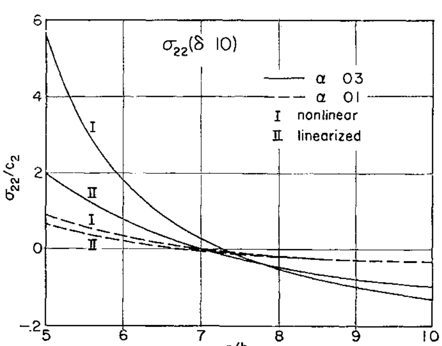

Fig 3 Distribution of tangential stress

The displacement gradients $\partial u/\partial r$ corresponding to Eq (15) are equal to 0 08 at $r = a$ and 0 024 at $r = b$ for $\alpha = 0\ 01$, and 0 33 at $r = a$ and 0 078 at $r = b$ for $\alpha = 0\ 03$

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Effect of Diameter upon Elastic Properties in Thin Fibers

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Introduction

THE use of thin fibers in composite materials is of considerable interest for a variety of applications Examples for the potential value of filamentary composites are the remarkable mechanical properties of filament-wound structures made from endless, thin glass fibers bonded with various organic resins These properties result, in part, from the size effect (increase in strength observed in thin glass fibers as compared to the bulk strength of glass), combined with an effective crack barrier function of the bonding material

Major limitations of those materials are their relatively low elastic modulus and their low temperature resistance

Considerable effort in government-sponsored research and development is expended at the present time to produce fibers of increased stiffness and temperature resistance Principal difficulties encountered in these efforts are 1) the formability of refractory materials is notably poor, and 2) there appears to be little hope for the development of an economical process producing large quantities of nonmetallic fibers of useful length from any other than "glassy" states of materials Glassy states, however, are normally of lower stiffness and temperature resistance than crystalline modifications of the same material Many refractories can be formed into glassy compounds (berillia, borides, etc) If the size effect in those materials can be controlled and exploited, thin fibers made of refractory "glasses" may be expected to provide significant advances in the state of the art of structural materials Thus, the penalties incurred in using glassy states for reasons of formability may be partially or fully compensated by the strengthening effect of small dimensions

A study of the size effects in glass fibers upon elastic (structure-dependent) properties appears worthwhile from two points of view: it may lead to a better understanding of the mechanisms that manifest themselves in the observed size effects on strength and temperature resistance in thin fibers, and the results of such a study may lead to useful guidelines in orienting the search for improved filamentary materials

Received December 13, 1963

* President

Table 1 Data on effective Young's modulus E and torsional modulus G

Diameter, μ	E kg/mm^2	G, kg/mm^2
100	4,900	2780
80	4,930	2870
60	5,030	2980
40	5,380	3080
20	6,320	3530
10	7,840	4240
5	9,490	5220
2	10,900	6750

Considerations of Size Effects upon Strength and Stiffness in Thin Fibers

A strong dependency of strength upon the thickness of test specimens has been established by numerous researchers for a variety of materials Griffith[1] and Jurkov[2] report maximum strengths of the order of 1 and 2 $\times$ 10^6 psi in fine glass and quartz fibers Beams[3] reports order-of-magnitude increases of strength in thin films of metals similar to those found by Brenner[4] and others in metallic and nonmetallic needle crystals ("whiskers")

Numerous theories have been postulated in attempts to explain the strengthening mechanisms of small dimensions Shanley,[5] for instance, shows that Griffith's experimental data can be well matched if it is assumed that a strong surface layer (skin) of approximately 2 μ thickness exists The measured strength is then a function of the proportions of cross-sectional area contributed by skin and core, respectively The existence of such a skin is plausible, since it must be assumed that surface layers will exhibit distortions in their atomic lattice because of the anomalies of interatomic forces present at the surface

Otto[6] has shown that, in the case of glass fibers, the observed increase in strength with decreasing fiber diameter is dependent upon the special circumstances attending their forming (temperature of melt, pulling speed, bushing orifice size), rather than upon the fiber diameter itself These finds do not, as one might conclude at first sight, disprove the concept of "strong skin," but rather they confirm the suspicion that the postulated "skin thickness," as well as possibly its physical properties, must be assumed to vary with variable temperature, speed, and geometry of the forming process Constant skin thickness in fibers of different diameters, then, could be attributed to a compensating effect of the various process parameter changes normally employed in the production of glass fibers with different thicknesses

A serious objection against the postulated "skin theory" is that, in order to explain the observed effects, the skin would either have to be prestressed at high stress levels or would have to exhibit a higher elastic modulus than the core, which in turn should result in a measurable dependency of the fiber's elastic properties upon diameter

It is not too well known that such a dependency has indeed been found in the case of fused quartz filaments Reinkober[7] reports the results of a series of carefully conducted experiments, showing an increase in both effective Young's modulus and torsional modulus with decreasing fiber diameter Table 1 shows data on effective Young's modulus E and torsional modulus G obtained by zone-averaging of a large number (approximately 20 for each point) of individual experiments

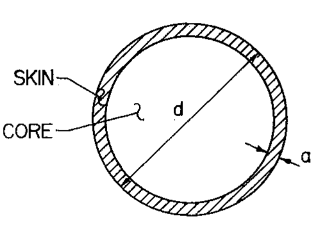

Fig 1 Skin-core hypothesis for fiber

Hypothetical Model of Elastic Fiber Properties

Following the suggestions of Shanley, it is assumed that the fiber is built up by a "skin" of constant thickness a surrounding a "core" of diameter $d - 2a$, as shown in Fig 1 Furthermore, assuming that the elastic properties of the skin are given by E and G and the elastic properties of the core by E and G, the effective moduli that would be observed in tension and torsion tests can be expressed as follows:

$$\bar{E} = E + 4(E - E)\, a/d - (a/d)^2 \tag{1}$$

$$\bar{G} = G + 8(G - G) \times [a/d - 3(a/d)^2 + 4(a/d)^3 - 2(a/d)^4] \tag{2}$$

Analysis of Data

The method of analysis employed is a least-error fitting of the experimental data listed in Table 1 with the analytical expressions for E and G given by Eqs (1) and (2), respectively

Since the hypothetical model shown in Fig 1 will obviously break down if the fiber diameter is less than twice the skin thickness, the experimental values for $d = 2\ \mu$ were discarded in the least-square fitting process Data obtained in this fashion are as follows:

1) Young's moduli and skin thickness for best fit of experimental E values:

$$E_s = 9682 \text{ kg/mm}^2$$
$$E = 4400 \text{ kg/mm}^2$$
$$a_E = 2\,02\ \mu$$

2) Torsion moduli and skin thickness for best fit of experimental G values:

$$G_s = 6080 \text{ kg/mm}^2$$
$$G = 2505 \text{ kg/mm}^2$$
$$a_G = 0\,735\ \mu$$

Comparison within the fitting interval of the computed and experimental values of fiber stiffness shows a maximum deviation of 1 06% in the case of Young's modulus E and 1 31% in the case of torsion modulus G These deviations are well within the experimental accuracy of the measured data

Discussion

The hypothetical model just postulated is admittedly very crude There is, for instance, no obvious reason why there should be a sharp transition between skin and core material, and why this transition should take place at a fixed depth a from the surface, particularly in view of Otto's data reported in Ref 6 Nevertheless, it appears to be significant that the hypothetical skin thickness for best fit falls into the order of magnitude in which strengthening effects in thin fibers, films, and whiskers have been observed for many materials It appears further significant that the hypothetical Young's modulus E of the skin assumes a value that is close to those found in crystalline bulk quartz ranging from 10,100 to 7750 kg/mm^2 in single crystals The ratios of E/G for both skin and core materials are smaller than 2, indicating that neither skin nor core is of isotropic material and further suggesting that the orientation of strongest bonds may be arranged along helical paths around the fiber axis

A somewhat disturbing factor is the difference in hypothetical skin thickness obtained from the two sets of elastic moduli It could be expected, however, that a more realistic "model" of the skin, e g , one involving a gradual transition from "skin" to "core" properties, would tend to reduce the discrepancy It should also be noted that two entirely different types of experiments were performed by Reinkober to establish the two sets of E and G values Young's modulus E was found by measuring elongations of the fibers subject to static loads; torsion modulus G was found by measuring the frequency of a torsion pendulum in which the test fiber provided the torsion spring Thus, a systematic discrepancy between the two sets could conceivably arise

Conclusions

The foregoing analysis of a limited set of experimental data tends to substantiate a hypothesis that the remarkable physical properties of thin glass fibers are associated with a skin layer of modified structure, posssibly exhibiting properties similar to those found in crystalline modifications of the same material Such a mechanism, if found effective in refractory materials capable of forming glassy compounds, would enhance the utility of glass-fiber-forming processes in the refractory materials field

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Calculation of the Surface Range of a Ballistic Missile

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A TECHNIQUE is developed which enables calculation of the distance between two points constrained to lie on the surface of an arbitrary ellipsoid The quadratic forms involved in the geometry are transformed in such a manner that integration of arc length may be computed by means of normal elliptical integrals of the second kind

Usually surface range is calculated on the surface of a sphere by the relation $s = r\theta$ where r = radius of sphere, and θ = subtended central angle (in radians) This range is, in effect, a great circle arc in a plane which passes through the geometrical center of the sphere The purpose of this note is to describe a technique for obtaining the arc range on the surface of an ellipsoid It is assumed that the definition of distance is taken to be the intersection of the given ellipsoid with a plane which passes through the geometrical center of the ellipsoid

When the burnout and impact subvehicle longitude and geocentric latitude of a missile are known,† denoted, respectively, by λ_{EB}, ϕ_B, and λ_{EI}, ϕ_I, the calculation of surface range can be accomplished in the following manner The coordinates of the burnout and impact points are given by

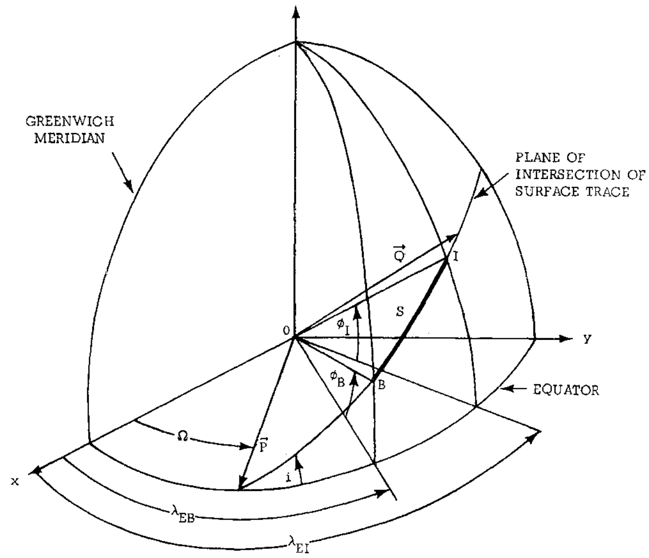

Fig 1 Coordinate system

$$x_B = r_B \cos\phi_B \cos\lambda_{EB} \qquad x_I = r_I \cos\phi_I \cos\lambda_{EI}$$
$$y_B = r_B \cos\phi_B \sin\lambda_{EB} \qquad y_I = r_I \cos\phi_I \sin\lambda_{EI} \qquad (1)$$
$$z_B = r_B \sin\phi_B \qquad z_I = r_I \sin\phi_I$$

Thus, the equation of plane BOI in Fig 1 is given at once by

$$\begin{vmatrix} x & y & z & 1 \\ x_B & y_B & z_B & 1 \\ x_I & y_I & z_I & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \qquad (2)$$

Letting the coefficients A, B, C be defined by Eq (2), it is possible to write

$$Ax + By + Cz = 0 \qquad (3)$$

where

$$A = y_B z_I - y_I z_B$$
$$B = x_I z_B - x_B z_I$$
$$C = x_B y_I - x_I y_B$$

The equation of an ellipsoid can be written as

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1 \qquad (4)$$

where a is the semimajor axis and c is the semiminor axis of the international ellipsoid Usually $a = b$, and an oblate spheriod is taken as the model of the earth The analysis, however, is not restricted to this case In canonical units a is usually taken to be unity, and c can be calculated from the relation

$$c = a(1 - f) \qquad (5)$$

where f is the flattening which is taken to be 1/298 3

By defining the unit vectors **P** and **Q**, (**P** ⊥ **Q**) (see Fig 1)

$$P_x = \cos\Omega \qquad Q_x = -\sin\Omega \cos i$$
$$P_y = \sin\Omega \qquad Q_y = \cos\Omega \cos i \qquad (6)$$
$$P_z = 0 \qquad Q_z = \sin i$$

where the orientation angles Ω and i are obtained from Ref 1 as

Received December 12, 1963 This work was supported by the Special Projects Office, Department of the Navy, under Contract No NOw 61 0145

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† The burnout and impact points are just two given arbitrary points It would be correct to insert the coordinates of New York and Paris and thus obtain the distance between the two cities